Year 5. Problem Set 119 (2009-2010 school year).

1. Suppose $P(x)$ is a polynomial. Prove that the remainder of the division of $P(x)$ by $x-c$ is $P(c)$.

2. What is the remainder of $x^{243}+x^{81}+x^{27}+x^{9}+x^{3}+x$ when divided by $x-1$.

3. $P(x)=x^{2n}+a^{2n}$. Prove that if $a\neq 0$ then $P(x)$ is not divisible by $x+a$ and is not divisible by $x-a$.

4. Prove that $a\frac{(x-b)(x-c)}{(a-b)(a-c)}+b\frac{(x-c)(x-a)}{(b-c)(b-a)}+c\frac{(x-a)(x-b)}{(c-a)(c-b)}=x$

5. Prove that the segments (0,1) and [0,1] have the same cardinality. Do not use any advanced theorems. Explicitly provide the mapping from one set to another.

6. An integer is called *formidable* if it can be written as a sum of distinct powers of 4, and *successful* if it can be written as a sum of distinct powers of 6. Can 2005 be written as a sum of a formidable number and a successful number? Prove your answer.

7. Prove that if two medians in a triangle are equal in length, then the triangle is isosceles.

8. There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.